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SURVEY OF INVESTIGATIONS ON THE ENERGY-MOMENTUM COMPLEX IN GENERAL RELATIVITY

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Synopsis

The paper contains a survey of the investigations of the last decade on the energy-momentum complex in general relativity. A comparison of the properties of the various complexes proposed in different papers is performed and their advantages and deficiencies are discussed. A satisfactory solution of the energy problem in accordance with the general principle of relativity has now been reached.

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Shortly after EINSTEIN had developed his final theory of gravitation in 1915 he also attacked the problem of energy and momentum conservation for the complete system of matter plus gravitational field. In his famous papers from the years 1915 and 1916 [1] he introduced the well-known expression for the energy-momentum complex

$$ {}_E\mathsf{T}_i^{\ k} = \mathfrak{T}_i^{\ k} + {}_E\tau_i^{\ k} \tag{1} $$

which satisfies the divergence relation

$$ {}_E\mathsf{T}_i^{\ k}{}_{,k} \equiv \frac{\partial\, {}_E\mathsf{T}_i^{\ k}}{\partial x^k} = 0 \tag{2} $$

as a consequence of the field equations. Here, $\mathfrak{T}_i^{\ k}$ is the matter tensor density, which appears as source of the gravitational field on the right-hand side of Einstein's field equations, while the gravitational part ${}_E\tau_i^{\ k}$ is a homogeneous quadratic expression in the first-order derivatives $g_{ik,l}$ of the metric tensor g_{ik}. In terms of the Einstein Lagrangian

$$ {}_E\mathfrak{L} = \sqrt{-g}\, g^{ik} (\Gamma^l_{ik} \Gamma^m_{lm} - \Gamma^l_{im} \Gamma^m_{kl}), \tag{3} $$

${}_E\tau_i^{\ k}$ has the canonical form

$$ {}_E\tau_i^{\ k} = \frac{1}{2\varkappa} \left(\frac{\partial\, {}_E\mathfrak{L}}{\partial g^{lm}{}_{,k}} g^{lm}{}_{,i} - \delta_i^k\, {}_E\mathfrak{L} \right) \tag{4} $$

where $\varkappa$ is Einstein's gravitational constant. ${}_E\mathfrak{L}$ is obtained from the scalar curvature density $\mathfrak{R}$ by omitting a divergence part containing the second order derivatives $g_{ik,l,m}$. It is an *affine* scalar density which is homogenously quadratic in the $g_{ik,l}$ and the expressions (1)–(4) can be obtained by the well-known method of (linear) infinitesimal coordinate transformations applied to ${}_E\mathfrak{L}$.

[1] This paper was reported at the Einstein Symposium der Deutschen Akademie der Wissenschaften, Berlin, in November 1965.

For a closed system and for a restricted class of coordinate systems the quantities obtained from ${}_E\mathfrak{T}_i^{\,k}$ by integrating over the spatial coordinates, i. e.

$$ {}_EP_i = \frac{1}{c}\iiint {}_E\mathsf{T}_i^{\,4}\, dx^1\, dx^2\, dx^3 \tag{5} $$

have quite remarkable properties. Before stating these properties we have to specify what we mean by a *closed* system. In general, an *insular* system, i. e. a system for which $\mathfrak{T}_i^{\,k}$ is zero outside a *time-like* tube of finite spatial extension, is not closed since it may lose energy by emission of gravitational radiation. This question has been studied extensively by Bondi et al [2] and by Sachs [3], and we can now give a general definition of a non-radiative system. A system is said to be closed if it is insular and, further, if it is possible to introduce a class of coordinates

$$ x^i = \{x, y, z, ct\},\; r = \sqrt{x^2 + y^2 + z^2} \tag{6} $$

with the following properties. Points at large spatial distances from the matter tube have large values of r, i. e. spatial infinity corresponds to $r \to \infty$. The metric is of the form

$$ g_{ik} = \eta_{ik} + \overset{(1)}{g}_{ik} \tag{7} $$

where η_{ik} is the constant Minkowski matrix and $\overset{(1)}{g}_{ik}$ and its first-order derivatives are asymptotically of the type

$$ \overset{(1)}{g}_{ik} = O_1, \quad g_{ik,l} = \overset{(1)}{g}_{ik,l} = O_2. \tag{8} $$

Here, O_n with positive integer n denotes a term for which $r^n\, O_n$ remains finite for $r \to \infty$. The coordinates defined by (6)–(8), the "B.S.-coordinates" for a closed system, are asymptotically Lorentzian since $g_{ik} \to \eta_{ik}$ for $r \to \infty$.

Now, by integrating (2) over a suitable cylindrical region of space-time and using Gauss's theorem one finds in a well-known way that the quantities ${}_EP_i$ have the following properties A, which are essential for the interpretation of P_i as the components of the four-momentum:

A *For a closed system and in a system of B.S.-coordinates the quantities P_i are constant in time and they transform as the components of a 4-vector under all linear transformations.*

These properties are contained in the more general statement, also following from (2), that the integrals

$$ {}_EP_i = -\frac{1}{c}\int_{\Sigma} {}_E\mathsf{T}_i^{\,k}\, dS_k \tag{9} $$

integrated over any space-like 3-dimensional hypersurface Σ of infinite extension are independent of the choice of Σ. For the validity of A it is essential that ${}_E\mathsf{T}_i^k$ is an affine tensor density of weight one and that the gravitational part ${}_E\tau_i^k$ is a homogeneous quadratic function of the $g_{ik,l}$, for this means that ${}_E\tau_i^k = O_4$ in a system of coordinates (6)–(8).

If we eliminate $\mathfrak{T}_i^k$ in (1) by means of the field equations the complex ${}_E\mathsf{T}_i^k$ appears as a function of the gravitational field variables for which the relation (2) must hold identically. This means that ${}_E\mathsf{T}_i^k$ may be written in the form

$$ {}_E\mathsf{T}_i^k = {}_E\psi_i^{kl}{}_{,l} \tag{10} $$

where $\psi_i^{kl} = -\psi_i^{lk}$, the so-called *superpotential,* is antisymmetrical in k and l. This possibility was first noted by VON FREUD [4], the explicit expression for ${}_E\psi_i^{kl}$ is [5]

$$ {}_E\psi_i^{kl} = \frac{g_{in}}{2\varkappa\sqrt{-g}}\,\mathfrak{g}^{nklm}{}_{,m} \tag{11} $$

with

$$ \mathfrak{g}^{iklm} = (-g)\,(g^{ik}g^{lm} - g^{il}g^{km}). \tag{12} $$

The latter quantity is a *true* tensor density of *weight two,* satisfying the symmetry relations

$$ \mathfrak{g}^{iklm} = -\mathfrak{g}^{ilkm} = -\mathfrak{g}^{mkli} = \mathfrak{g}^{kiml}, \tag{13} $$

while ${}_E\psi_i^{kl}$, which is a homogeneous linear function of the $g_{ik,l}$, is an *affine* tensor density of *weight one.*

By means of Stoke's theorem one gets from (9) and (10) for the four-momentum

$$ {}_EP_i = -\frac{1}{2c}\int_{\Phi} {}_E\psi_i^{kl}dS_{kl} \tag{14} $$

where the integration is extended over the 2-dimensional boundary surface Φ of Σ corresponding to a large constant value r_1 of the "radius" r (strictly speaking one has to take the limit $r_1 \to \infty$). Thus, ${}_EP_i$ depends only on the asymptotic values of the metric and it is, therefore, invariant under all coordinate transformations which preserve the asymptotic form of g_{ik}.

By means of (1) the equations (10) may be written

$$ {}_E\psi_i^{kl}{}_{,l} - {}_E\tau_i^k = \mathfrak{T}_i^k \tag{15} $$

which obviously is a special form of Einstein's field equations. If we raise the index i by means of the metric tensor g^{ik} these equations can also be brought into the form

$$_B\psi^{ikl}{}_{,l} - {}_B\tau^{ik} = \mathfrak{T}^{ik} \tag{16}$$

where

$$_B\psi^{ikl} \equiv \frac{1}{2\varkappa\sqrt{-g}}\,\mathfrak{g}^{iklm}{}_{,m} = -{}_B\psi^{ilk} \tag{17}$$

and $_B\tau^{ik}$ again is a homogeneous quadratic function of the $g_{ik,l}$. In this way we arrive at the complex first given by Bergmann and Thomson [6].

$$_B\mathsf{T}^{ik} \equiv \mathfrak{T}_i^{\,k} + {}_B\tau^{ik} = {}_B\psi^{ikl}{}_{,l}. \tag{18}$$

The integrated quantities $_BP^i$ obtained from this complex in a similar way as $_EP_i$ in (5) or (9) also have the properties A. Moreover, in any system of B.S.-coordinates we have simply

$$_EP_i = \eta_{ik}\,{}_BP^k \tag{19}$$

i. e. the two different complexes give the same values for the total momentum and energy in such coordinates.

Similar properties has the following complex given by Landau and Lifshitz [6]:

$$_L\mathsf{T}^{ik} = {}_L\psi^{ikl}{}_{,l} \tag{20}$$

$$_L\psi^{ikl} = \frac{1}{2\varkappa}\,\mathfrak{g}^{iklm}{}_{,m}\,. \tag{21}$$

From (16)–(21) it follows that

$$_L\mathsf{T}^{ik} = \sqrt{-g}\,(\mathfrak{T}^{ik} + {}_L\tau^{ik}) \tag{22}$$

where $_L\tau^{ik}$ like $_B\tau^{ik}$ is an affine tensor density of weight one, which is a homogeneous quadratic function of the $g_{ik,l}$. Consequently $_L\mathsf{T}^{ik}$ is an affine tensor density of weight two, which means that $_LP^i$ is a 4-vector under Lorentz transformations only. On the other hand $_L\mathsf{T}^{ik}$ has the advantage of being symmetrical in i and k as is seen at once from (20), (21) and (13). In any system of B.S.-coordinates we have

$$_LP^i = {}_BP^i = \eta^{ik}\,{}_EP_k \tag{23}$$

so that the three different complexes are equally suited for the calculation of the four-momentum in such coordinates. However, in more general systems of coordinates the application of these complexes leads to meaningless results. From the point of view of general relativity this is not satisfactory and in the past this has caused some doubts about the applicability of these complexes at all. As a matter of fact we must require of a truly generally relativistic expression for the four-momentum that it satisfies the following condition:

B For a closed physical system the total four-momentum is a free 4-vector under arbitrary space-time transformations.

The necessity for this requirement is seen at once if we go to the limit of spatially very small systems, for in this case our system is effectively a particle which, according to basic assumptions of general relativity, certainly should have a four-momentum with this property.

A somewhat weaker requirement contained in B is the condition that

B′ the fourth component of the four-momentum must be invariant under purely spatial transformations

$$x'^{\iota} = f^{\iota}(x^{\varkappa}), \; x'^4 = x^4 \tag{24}$$

i. e.

$$P'_4 = P_4 \tag{25}$$

which expresses the physically evident fact that the total energy is invariant under such transformations.

Now, none of the forementioned complexes satisfy even this rather weak and triviel condition. In the case of the Einstein complex this was pointed out first by Bauer [7] who remarked that in a completely empty space Einstein's expression for the total energy gives the correct value zero in a Cartesian system of coordinates, but the meaningless value $-\infty$ when calculated in polar coordinates. For this reason the whole question of the energy in gravitational fields was taken up again in 1958 [5], and it was shown that it is possible to define a complex

$$\left.\begin{aligned} \Theta_i^{\,k} &= \mathfrak{T}_i^{\,k} + \vartheta_i^{\,k} = \chi_i^{\,kl}{}_{,l} \\ \chi_i^{\,kl} &= \frac{\sqrt{-g}}{\varkappa}\,(g_{in,m} - g_{im,n})\, g^{km} g^{ln} \end{aligned}\right\} \tag{26}$$

which satisfies the condition B'. In fact it follows from (26) that $\Theta_4^{\,4}$ is a scalar density under purely spatial transformations which means that the Bauer difficulty does not arise with this complex. Further, it seemed that this complex made it possible to give an unambiguous meaning to the distribution of the energy throughout space-time. Similarly as the Einstein expression is obtained from the Lagrangian ${}_{E}\mathfrak{L}$, the complex (26) follows by the method of infinitesimal coordinate transformations applied to the complete scalar curvature density $\mathfrak{R}$ [8]. However, a closer consideration showed that the complex (26) does not satisfy the condition *A*. This is connected with the fact that $\vartheta_i^{\ k}$ besides the $g_{ik,l}$ also contains the second-order derivatives $g_{ik,l,m}$. Furthermore, it is not sufficient to consider the energy only, i. e. besides the condition B' we have to require the full condition *B* to be satisfied and this is not the case either for the complex (26). In fact the applicability of the latter complex is even more restricted than the three former complexes.

In a recent paper, which will appear in the Report of the Conference on Elementary Particles held in Kyoto in September 1965, the question was discussed what properties of the energy-momentum complex $\mathsf{T}_i^{\ k}$ are necessary and sufficient in order that the integrated quantities P_i have all the properties *A* and *B*. The result was the following:

1.
$$\mathsf{T}_i^{\ k} = \mathfrak{T}_i^{\ k} + \tau_i^{\ k} \tag{27}$$

is an affine tensor density of weight one satisfying the relation

$$\mathsf{T}_{i\ ,k}^{\ k} = 0 . \tag{28}$$

in every system of space-time coordinates.

2. $\tau_i^{\ k}$ *is a function of the gravitational field variables which, in a B.S.-system of coordinates* (6)–(8) *for a closed system, satisfies the relation*

$$r^3\, \tau_i^{\ k} \to 0 \ \text{ for } \ r \to \infty . \tag{29}$$

3. The superpotential $\mathfrak{U}_i^{\ kl} = -\mathfrak{U}_i^{\ lk}$, *which expresses* $\mathsf{T}_i^{\ k}$ *in the form*

$$\mathsf{T}_i^{\ k} = \mathfrak{U}_{i\ \ ,l}^{\ kl} \tag{30}$$

is a true tensor density depending on the gravitational field variables and their first-order derivatives only.

The conditions *1* and *2* ensure that the integrated quantities

$$P_i = \frac{1}{c}\iiint \mathsf{T}_i^{\ 4}\, dx^1\, dx^2\, dx^3 = -\frac{1}{c}\int\limits_{\Sigma} \mathsf{T}_i^{\ k}\, dS_k = -\frac{1}{2\,c}\int\limits_{\Phi} \mathfrak{U}_i^{\ kl}\, dS_{kl} \tag{31}$$

have the properties *A*. Further, with the assumption *3.* the quantity $dA_i = \mathfrak{U}_i^{\ kl}\, dS_{kl}$ is a true 4-vector on Φ. Therefore, since space-time for a closed system can be treated as flat on and outside Φ the vectors dA_i can in a unique way be parallel-displaced to a common point P on or outside Φ so as to form a true 4-vector at the point P. Thus P_i is a true free 4-vector. It should be noted that, for a system with sufficiently small spatial extension say an atomic system, "spatial infinity" is practically reached already at very small distances, so that the "radius" r_1 of Φ in such cases may even be taken microscopically small.

None of the complexes mentioned so far satisfy the condition *3*. In fact it is evident that no complex containing the metric tensor only can satisfy this condition, for it is impossible to construct a true tensor density $\mathfrak{U}_i^{kl}$ out of g_{ik} and its first-order derivatives. This shows that one has to introduce a new element into the space-time manifold of general relativity and this can be done in different ways.

Following ideas of Rosen [9], Cornish [10] introduces a flat space metric $\overset{(o)}{g}_{ik}$ which asymptotically for large spatial distances agrees with g_{ik}. The mapping of the real space-time with metric g_{ik} on the imaginary flat space-time with the metric $\overset{(o)}{g}_{ik}$ may for instance be performed by assuming that $\overset{(o)}{g}_{ik}$ in a *definite* B.S.-system of coordinates (6)–(8) has components $\overset{(o)}{g}_{ik} = \eta_{ik}$ throughout space-time. In any other system of coordinates obtained by a non-linear transformation the components of $\overset{(o)}{g}_{ik}$ are then not constant although, of course, the curvature tensor corresponding to the metric $\overset{(o)}{g}_{ik}$ vanishes in all systems. Now, if the covariant derivative of a tensor A_{ik} corresponding to the metric $\overset{(o)}{g}_{ik}$ is denoted by $A_{ik/l}$ one may, starting from ${}_E\psi_i{}^{kl}$ in (11), define a superpotential

$$ {}_C\mathfrak{U}_i^{kl} = \frac{g_{in}}{2\varkappa\sqrt{-g}}\,\mathfrak{g}^{nklm}{}_{/m} \tag{32} $$

which obviously is a tensor density under arbitrary space-time transformations. Then, the complex

$$ {}_C\mathfrak{T}_i^{k} = {}_C\mathfrak{U}_i^{kl}{}_{,l} \tag{33} $$

satisfies all the conditions *1–3* and the corresponding integrated quantities ${}_CP_i$ will have all the properties A and B. In a similar way one could start from the superpotentials ${}_B\psi^{ikl}$ and ${}_L\psi^{ikl}$ and construct true tensor densities by means of the flat space metric $\overset{(o)}{g}_{ik}$. In this way one would arrive at two other expressions for the total four-mementum which are numerically identical with the one following from (32), (33).

However, this method of obtaining true tensor densities by introducing an unobservable metric does not seem to me quite satisfactory. Apart from the arbitrariness in the mapping of the real space-time on the imaginary flat space-time which perhaps is not so serious since it does not effect the values of the total four-momentum, the introduction of a metric $\overset{(o)}{g}_{ik}$ which to a large extent is independent of the observable metric g_{ik} makes the covariance obtained rather formal and deprives the general principle of relativity of its physical content. If one introduces unobservable quantities, they should rather be of a similar type as the potentials in electrodynamics from which the observable quantities, in our case the g_{ik}, can be calculated uniquely. As was shown in a series of recent papers [11–13] it is, in fact, possible to obtain a satisfactory expression for the energy-momentum complex, satisfying all requirements, if one describes the gravitational field by means of tetrads $h_i^{(a)}$ which uniquely determine the metric tensor by the equations

$$g_{ik} = h_i^{(a)} h_{(a)k} \,. \tag{34}$$

Here, the index (a), which is raised and lowered by means of the constant Minkowski matrix $\eta_{(ab)} = \eta^{(ab)}$ numbers the four tetrad vectors $h_i^{(a)}(x)$ at the arbitrary point (x). The use of tetrads to describe the gravitational field is by no means new. In fact, tetrads enter as an essential element in the generally relativistic formulation of the Dirac equations for Fermion fields.

If one eliminates g_{ik} in the expression for the scalar curvature density $\mathfrak{R}$, by means of (34), $\mathfrak{R}$ appears as the sum of a divergence part and a new Lagrangian $\mathfrak{L}$ which is a homogeneous quadratic expression in the first-order derivatives $h_{i,k}^{(a)}$ of the tetrad variables. The explicit expression is [11]

$$\mathfrak{L} = \sqrt{-g}\,[\gamma_{rst}\,\gamma^{tsr} - \Phi_r\,\Phi^r] \tag{35}$$

where γ_{ikl} and Φ_k are the following true tensor and vector, respectively,

$$\gamma_{ikl} = h_i^{(a)}\,h_{(a)k;l}\,, \quad \Phi_k = \gamma^i{}_{ki} \tag{36}$$

Here, the semicolon means covariant derivation corresponding to the real observable metric g_{ik}. Thus, in contrast to the Einstein Lagrangian $\mathfrak{L}$ in (3) the Lagrangian $\mathfrak{L}$ is a true scalar density. If we apply the method of arbitrary infinitesimal coordinate transformations to this Lagrangian $\mathfrak{L}$ we get an energy-momentum complex (27) satisfying the condition *1*. Further

$$\tau_i^{\ k} = \frac{1}{2\varkappa}\left[\frac{\partial \mathfrak{L}}{\partial h^{(a)}_{l,k}} h^{(a)}_{l,i} - \delta_i^{\ k}\,\mathfrak{L}\right] \tag{37^1}$$

is a homogeneous quadratic function of the $h^{(a)}_{i,k}$ which is essential for the validity of *2*. The corresponding superpotential is

$$\mathfrak{U}_i^{\ kl} = -\mathfrak{U}_i^{\ lk} = \frac{\sqrt{-g}}{\varkappa}\left[\gamma^{kl}{}_i - \delta_i^k \Phi^l + \delta_i^l \Phi^k\right] \tag{37}$$

which shows that also the condition *3*. is satisfied. Thus, the complex following from (37) by (30) satisfies all the conditions *1–3* and our problem seems to be solved.

However, the tetrad field $h_i^{(a)}$ is not determined uniquely by (34) for a given metric field g_{ik}. In fact, any Lorentz rotation of the tetrads,

$$\check{h}_i^{(a)} = \Omega^{(a)}{}_{(b)}(x)\, h_i^{(b)} \tag{38}$$

leaves the right-hand side of (34) unchanged. Here, the rotation coefficients $\Omega^{(a)}{}_{(b)}(x)$ may be any scalar functions of (x) which satisfy the orthogonality relations at each point, and the complex $\mathsf{T}_i^{\ k}$ obtained from the superpotential (37) is *not* invariant under the "gauge" transformations (38), except if the rotation coefficients are constants throughout space-time. For an arbitrary physical system there are no physically convincing arguments for fixing the gauge so as to make $\mathsf{T}_i^{\ k}(x)$ a unique function of the space-time coordinates, but in the case of a completely empty flat space there is no doubt about the choice of the tetrads. In order to avoid the forementioned Bauer difficulty it is necessary in that case to require that the tetrad field forms a system of mutually parallel tetrads throughout space-time, i. e. we must have everywhere

$$h^{(a)}_{i;k} = 0. \tag{39}$$

Further, for an insular system, where space-time is asymptotically flat we must require that the tetrad fields at least *asymptotically* form a system of parallel vectors. This suggests that the tetrads in a system of B.S.-coordinates must satisfy the same boundary conditions as the metric at large spatial distances. For a closed system this would mean relations analogous to (7), (8), i. e.

$$\left.\begin{aligned} h_{(a)i} &= \eta_{ai} + \overset{(1)}{h}_{ai} \\ \overset{(1)}{h}_{ai} &= O_1, \ \overset{(1)}{h}_{ai,k} = O_2 \end{aligned}\right\} \tag{40}$$

Then, as was shown in reference [12] for an even more general system emitting gravitational radiation, the total four-momentum P_i is invariant under all gauge transformations (38) which respect the boundary conditions. Besides, of course, P_i is invariant under all Lorentz rotations of the tetrads with constant rotation coefficients $\Omega^{(a)}{}_{(b)}$. On the other hand, the complex $\mathsf{T}_i^{\,k}$ itself is invariant only under the latter type of gauge transformations. Therefore, unless one can find a good physical argument for fixing the gauge throughout the system, it has no physical meaning to speak about the energy distribution inside the system. This would be in complete agreement with Einstein's own point of view. Actually nobody has so far been able to give a prescription for measuring the energy of the gravitational field in a small region, in contrast to the total energy for which such prescriptions are easily given [13].

In any system of B.S.-coordinates, the values of P_i obtained from the tetrad complex (37) are the same as those obtained from the metric complexes (11), (17), (21) of Einstein, Bergmann and Landau. Therefore, once the generally covariant expression of P_i has been established by way of the tetrad formalism, we may forget about the tetrads and perform the calculation of P_i in a system of coordinates in which the purely metric-dependent complexes are known to be valid. Then, the values of P_i in an arbitrary system of coordinates can be obtained by using the law of transformation of a 4-vector.

Anyhow, the tetrad formulation has given us more confidence in the application of the energy-momentum complexes which for many years by many physicists have been regarded as not quite respectable quantities. We are also encouraged to apply them to more general physical systems. Up till now we have only considered the case where space time far away from our system is flat. What about a system in a permanent external gravitational field, for instance a planet in the field of a heavy central body like the sun? If the external gravitational field is practically constant over a region of extension l large compared with the dimensions of the planet the preceding considerations are easily generalized. We have only to choose the tetrads of the external field so that the equation (39) is satisfied at each point of the time-track of the planet. This can always be obtained by a suitable transformation (38). In a system of Fermi-coordinates where the external metric has vanishing first-order derivatives at all points of the time-track of the planet we then get by integrating $\mathsf{T}_i^{\,4}$ over a sphere enclosing the planet but with a radius smaller than l a four-momentum P_i^{pl} for the planet which is a 4-vector in the space-time with the external metric.

The energy-momentum complexes can also be used for calculating the total energy and momentum for systems which emit gravitational radiation in which case these quantities are not constant of course. Also the amount of energy and momentum emitted in different directions can be calculated. Such calculations were performed in reference [12]. As regards the total energy and its variation in time the results obtained are in agreement with and corroborate earlier results of Bondi [2] and Sachs [3].

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